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The plane contact problem for a prestressed incompressible elastic layer☆

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ABSTRACT

The problem of the indentation of a rigid punch into the upper face of a layer when a uniform field of initial stresses is present in the layer is considered. A model of an isotropic incompressible non-linearly elastic material, specified by the Mooney elastic potential, is used. The case when the layer rests on the lower face without friction is investigated. It is assumed that the additional stresses, due to the punch indentation, are small compared with the initial stresses. This assumption enables the problem of determining the initial stresses to be linearized. It is later reduced to the solution of an integral equation of the first kind with a difference kernel with respect to the pressure in the contact region. Depending on the dimensionless parameter λ, characterizing the relative thickness of the layer, asymptotic solutions are constructed for large and small values of this parameter. A solution for a whole range of values of the parameter, investigated by the "large" and "small" λ methods, is also obtained using a modified Multhopp–Kalandiya method.

1. Formulation of the problem

Consider a layer of thickness h of isotropic incompressible non-linearly elastic material, resting on the lower face without friction. The layer is in a uniform stress state, produced by stretching forces, applied at infinity. We will choose a system of coordinates Oxy such that the Oy axis is perpendicular to the layer surface. Then, the components of the stress tensor in the initial state have the form

$$\sigma_{11}^0 = s, \quad \sigma_{22}^0 = \sigma_{33}^0 = \sigma_{12}^0 = \sigma_{13}^0 = \sigma_{23}^0 = 0 \tag{1.1}$$

We will further assume that, after preliminary large deformation, a rigid punch, having the shape of an infinite strip

$$|x| \le a, \quad -\infty < y < \infty$$

acts on the upper face of the layer.

We will assume that the perturbations of the strains and stresses produced by the action of the punch are relatively small. This enables us to linearize the problem of determining the additional stresses and displacements on the background of the main stress-strain state.

We will assume that the elastic properties of the material are specified by the Mooney elastic potential[1] [1]. Hence, the components of the tensor of the additional stresses after linearization will have the form

$$\sigma_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x}\right), \quad \sigma_{yy} = 2\mu\frac{\partial \upsilon}{\partial y} + q \tag{1.2}$$

where q is the additional hydrostatic pressure and u and v are additional displacements along the x and y axes. Taking this into account, we obtain Lamé's equations. Supplementing them by the condition of incompressibility, we have a complete system of equations for determining the unknown displacements u and v and the function q

$$\mu\Delta u - s\frac{\partial^2 u}{\partial x^2} + \frac{\partial q}{\partial x} = 0, \quad \mu\Delta \upsilon - s\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial q}{\partial y} = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} = 0 \tag{1.3}$$

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The boundary conditions have the form

$$y = h : \sigma_{yx} = 0; \quad \sigma_{yy} = 0 \text{ when } |x| > a, \quad \upsilon = -[\delta + \beta x - f(x)] \text{ when } |x| \le a$$
$$y = 0 : \sigma_{yx} = 0, \quad \upsilon = 0 \qquad (1.4)$$

Here δ is the depth of the punch into the layer, β is its angle of rotation around the axis perpendicular to the Oxy plane, and $f(x)$ specifies the punch surface.

It is necessary to consider an auxiliary problem in which we add one more boundary condition to those already mentioned, namely, with respect to the normal stress under the punch:

$$y = h, \quad |x| \le a: \quad \sigma_{yy} = -q_1(x)$$

We will show later that the problem reduces to determining this contact pressure from the integral equation.

2. Reduction of the problem to the solution of an integral equation

We will apply a Fourier transformation to Eq. (1.3) and, using its properties, we arrive at a system of ordinary differential equations in the transformants, for which we introduce the notation

$$U(\alpha,y) = \int_{-\infty}^{\infty} u(x,y)e^{i\alpha x}\,dx, \quad V(\alpha,y) = \int_{-\infty}^{\infty} \upsilon(x,y)e^{i\alpha x}\,dx, \quad Q(\alpha,y) = \int_{-\infty}^{\infty} q(x,y)e^{i\alpha x}\,dx$$

The general solution then has the form

$$V = (C_1 + C_2\alpha y)e^{\alpha y} + (C_3 + C_4\alpha y)e^{-\alpha y}$$
$$U = (C_3 - C_4(1-\alpha y))e^{-\alpha y} - i(C_1 + C_2(1+\alpha y))e^{\alpha y}$$
$$Q = \alpha(sC_3 - C_4(s + 2\mu - s\alpha y))e^{-\alpha y} - \alpha(sC_1 + C_2(s + 2\mu + s\alpha y))e^{\alpha y} \qquad (2.1)$$

In order to determine the unknown functions $C_i = C_i(\alpha)$ ($i = 1, 2, 3, 4$), we substitute solutions (2.1) into boundary conditions (1.4), transformed into boundary conditions for the transformants. We also introduce the following notation for the transformant of the contact pressure

$$Q_1(\alpha) = \int_{-a}^{a} q_1(\xi)e^{i\alpha\xi}d\xi$$

As a result we arrive at a system four linear algebraic equations in $C_i(\alpha)$, solving which, we obtain, in particular,

$$\upsilon(x,h) = -\frac{1}{4\pi\mu}\int_{-\infty}^{\infty} Q_1(\alpha)\frac{L(\alpha h)}{\alpha}e^{-i\alpha x}d\alpha; \quad L(u) = \frac{2(\mathrm{ch}^2 u - 1)}{2\mathrm{ch}u\,\mathrm{sh}u + 2u - s\mu^{-1}u}$$

We will use the last boundary condition (the other ones are already satisfied). We obtain

$$\upsilon(x,h) = -[\delta + \beta x - f(x)] \equiv -g(x), \quad |x| \le a$$

Note that the function $K(u) = L(u)/u$ is even, and hence, using the representation for the transformant $Q_1(\alpha)$, we arrive at an integral equation, which, after changing to the dimensionless quantities

$$x' = \frac{x}{a}, \quad \xi' = \frac{\xi}{a}, \quad \lambda = \frac{h}{a}, \quad s' = \frac{s}{\mu}, \quad \varphi(\xi') = \frac{q_1(\xi)}{\mu}, \quad \tilde{g}(x') = \frac{2g(x)}{a}$$

(henceforth the primes and tildes will be omitted) can be represented as follows:

$$\int_{-1}^{1}\varphi(\xi)d\xi\int_{0}^{\infty}\frac{L(u)}{u}\cos\frac{u(\xi - x)}{\lambda}du = \pi g(x), \quad |x| \le 1 \qquad (2.2)$$

3. The modified Multhopp–Kalandiya method[2]

This method enables us, using a certain discretization of the integral equation, to reduce it to a system of linear algebraic equations. We will have in mind the behaviour of the function $L(u)$ at zero and at an infinity

$$L(u) = \frac{2}{4-s}u + O(u^3), \quad u \to 0; \quad L(u) = 1 + O(e^{-2u}), \quad u \to \infty$$

It can be shown that in this case the kernel of integral Eq. (2.2) enables us to isolate the logarithmic singularity

$$k(t) = -\ln|t| + F(t) \qquad (3.1)$$

where $F(t)$ is a regular even function. Consequently, we can reduce Eq. (2.2) to the form

$$-\int_{-1}^{1}\varphi(\xi)\ln\left|\frac{\xi-x}{\lambda}\right|d\xi=\pi g(x)-\int_{-1}^{1}\varphi(\xi)F\left(\frac{\xi-x}{\lambda}\right)d\xi,\quad |x|\le 1 \tag{3.2}$$

Suppose the function $g(x)$ is such that its derivative satisfies the Lipschitz condition

$$|g'(x_1)-g'(x_2)|\le m|x_1-x_2|,\quad \forall x_1,\quad x_2\in[-1,1];\quad m=\text{const}$$

Then, with this value of λ a solution of Eq. (3.2) exists in the class L_p $[-1, 1]$ $(1<p<2)$, and it can be represented in the form

$$\varphi(x)=\Phi(x)/\sqrt{1-x^2}$$

where the function $\Phi(x)$ is at least continuous.

We will construct an interpolation Lagrange polynomial $\tilde{\Phi}(\theta)$ $(x=\cos\theta)$ for it with Chebyshev nodes. Then, after replacing the function $\Phi(x)$ in Eq. (3.2) by its Lagrange polynomial, the integrals on the left-hand side of this equation can be calculated explicitly, while on the right-hand side, with integrand $F(t)$, they can be calculated approximately using Gauss quadrature formula. It was also noted previously that the function $F(t)$ is even, and hence we can take an odd number of nodes $N=2l+1$ and obtain a system of $l+1$ linear algebraic equations for determining the quantities $\Phi(\theta_n)$ $(n=1,\ldots,l+1;\ \theta_n=\pi(2n-1)/2N,\ x_n=\cos\theta_n)$

$$\sum_{n=1}^{l+1}\tilde{\Phi}(\vartheta_n)\delta_n\left\{\ln 2\lambda+\Psi_l^+(\vartheta_n,\vartheta_k)+\frac{1}{2}\left[F\left(\frac{\cos\vartheta_n-\cos\vartheta_k}{\lambda}\right)+\right.\right.$$
$$\left.\left.+F\left(\frac{\cos\vartheta_n+\cos\vartheta_k}{\lambda}\right)\right]\right\}=\left(l+\frac{1}{2}\right)\hat{g}(\vartheta_k),\quad k=1,\ldots,l+1$$

where

$$\Psi_l^+(\psi,\vartheta)=\sum_{m=1}^{l}\frac{\cos 2m\psi\cos 2m\vartheta}{m},\quad \hat{g}(\vartheta_k)=g(x_k),\quad \delta_n=\begin{cases}1, & n\ne l+1\\ 1/2, & n=l+1\end{cases}$$

4. Asymptotic solution for a large relative thickness of the layer[3]

To construct the asymptotic solution, we will use representations (3.1) and (3.2), and we will expand the function $F(t)$ in a converging power series, taking into account the fact that

$$k(t)=-\ln|t|-\sum_{n=0}^{\infty}a_nt^{2n}$$

$$a_0=\int_0^{\infty}\frac{1-L(u)-e^{-u}}{u}du,\quad a_n=\frac{(-1)^n}{(2n)!}\int_0^{\infty}(1-L(u))u^{2n-1}du,\quad n=1,2,\ldots \tag{4.1}$$

Consequently, Eq. (3.2) becomes

$$-\int_{-1}^{1}\varphi(\xi)\ln\left|\frac{\xi-x}{\lambda}\right|d\xi=\pi g(x)+\sum_{n=0}^{\infty}\frac{a_n}{\lambda^{2n}}\int_{-1}^{1}\varphi(\xi)(\xi-x)^{2n}d\xi,\quad |x|\le 1 \tag{4.2}$$

We will write the solution of Eq. (4.2) in the form of a series

$$\varphi(x)=\sum_{n=0}^{\infty}\frac{\varphi_n(x)}{\lambda^{2n}} \tag{4.3}$$

Substituting expression (4.3) into Eq. (4.2) and equating terms of like powers of λ, we arrive at an infinite system of successively solvable integral equations in the function ϕ_n. The solution has the form

$$\varphi(x)=\frac{N_0}{\pi\sqrt{1-x^2}}\left[1-\frac{2a_1}{\lambda^2}\left(\frac{1}{2}-x^2\right)-\frac{4a_2}{\lambda^4}\left(\frac{7}{8}-x^2-x^4\right)\right]-$$
$$-\frac{1}{\pi\sqrt{1-x^2}}\int_{-1}^{1}\sqrt{1-\xi^2}g'(\xi)\left\{\frac{1}{\xi-x}+\frac{2a_1x}{\lambda^2}+\frac{1}{\lambda^4}[2a_2(6x^3-6x^2\xi+\right.$$
$$\left.+2x\xi^2-2x+3\xi)+2a_1^2x]\right\}d\xi+O\left(\frac{1}{\lambda^6\sqrt{1-x^2}}\right);\quad N_0=\int_{-1}^{1}\varphi(\xi)d\xi$$

where N_0 is the dimensionless pressure under the punch, given by the equation

$$N_0 = \left[\ln 2\lambda - a_0 - \frac{a_1}{\lambda^2} - \frac{a_1^2}{4\lambda^4} - \frac{9a_2}{4\lambda^4} + O(\lambda^{-6})\right]^{-1} \times$$
$$\times\left[\int_{-1}^{1}\frac{\hat{g}(\xi)d\xi}{\sqrt{1-\xi^2}} + \frac{1}{\lambda^2}\int_{-1}^{1}\sqrt{1-\xi^2}\hat{g}(\xi)\xi\left(a_1 + \frac{a_2\xi^2}{\lambda^2} + \frac{7a_2}{2\lambda^2}\right)d\xi + O(\lambda^{-6})\right]$$

s	Multhopp–Kalandiya method				Asymptotic solution				
	$\lambda = 0.5$	1	2	4	0.5	1	2	2	4
0	17.42	9.42	5.48	3.57	16.68	9.21	5.47	5.57	3.48
2	9.77	5.76	3.75	2.69	9.69	5.74	3.78	3.76	2.69
3	5.60	3.60	2.58	2.00	5.49	3.57	2.67	2.58	2.00

5. Asymptotic solution for a small relative thickness of the layer[3]

It can be shown that, for sufficiently small λ, the solution of Eq. (2.2) can be sought in the form

$$\varphi(x) = \omega^{(1)}\left(\frac{1+x}{\lambda}\right) + \omega^{(2)}\left(\frac{1-x}{\lambda}\right) - \omega^{(0)}\left(\frac{x}{\lambda}\right) \tag{5.1}$$

The functions $\omega^{(1)}$, $\omega^{(2)}$, $\omega^{(0)}$ are solutions of the integral equations

$$\begin{gathered}\int_0^\infty \omega^{(1)}(\tau)k(\tau-t)d\tau = \frac{\pi}{\lambda}g(\lambda t - 1), \quad 0 \le t < \infty \\ \int_0^\infty \omega^{(2)}(\tau)k(\tau-t)d\tau = \frac{\pi}{\lambda}g(1 - \lambda t), \quad 0 \le t < \infty \\ \int_0^\infty \omega^{(0)}(\tau)k(\tau-t)d\tau = \frac{\pi}{\lambda}g(\lambda t), \quad -\infty < t < \infty\end{gathered} \tag{5.2}$$

The solutions of the first two equations of (5.2) can be found by the Wiener–Hopf method,[4] and the solution of the last one can be found using the convolution theorem for the Fourier integral transformation.

We will consider a specific simple case. Suppose $g(x) = \delta' = 2\delta/a$ (the prime will henceforth be omitted). Then the solution of the third integral equation of (5.2) has the form

$$\omega^{(0)}(x) = \frac{(4-s)\delta}{2\lambda} \tag{5.3}$$

To construct solutions of the first two integral equations of (5.2) in analytic form we approximate the function $L(u)/u$ by the expression

$$\frac{L^*(u)}{u} = \frac{\sqrt{u^2+B^2}}{u^2+A^2} \tag{5.4}$$

The constants A and B are chosen so as to satisfy the relations

$$L^*(u) = \frac{2}{4-s}u + O(u^3), u \to 0; \quad L^*(u) = 1 + O\left(\frac{1}{u^2}\right), u \to \infty$$

Taking the approximation into account we obtain

$$\omega^{(1)}(x) = \omega^{(2)}(x) = \frac{\delta A}{\lambda\sqrt{B}}\psi(x), \quad \psi(x) = \frac{A}{\sqrt{B}}\operatorname{erf}(\sqrt{Bx}) + \frac{e^{-Bx}}{\sqrt{\pi x}}$$

Substituting the solutions obtained into (5.1) and evaluating the integrals we obtain

$$N_0 = \delta\left[\left(\frac{A}{B} - 1\right)^2\left(e^{-2B/\lambda} - 1\right) + \frac{2A^2}{\lambda B} + 1\right]$$

The table shows values of N_0 for different λ and s. For each value of s the approximation $L^*(u)$ was constructed so that the error did not exceed 14%.

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